MoPAT24 Moments and Polynomials: Applications and Theory

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Sunrise over Lake Konstanz (by Philipp di Dio, 9th of November 2023, 7:02)

The interplay and connection between the theory of moments and polynomials was always strong, almost inseparable, and probably no theory would be able to evolve without the other.

In this conference we want to bring both aspects together again by looking at past, present, and hopefully future developments. The topics shall not only cover the classical theory of moments and polynomials, but also look into matrix moments and polynomials as well as applications.

Organized by Philipp di Dio and Tobias Sutter.

Sunrise over Lake Konstanz (by Philipp di Dio, 14th of August 2023, 6:32)

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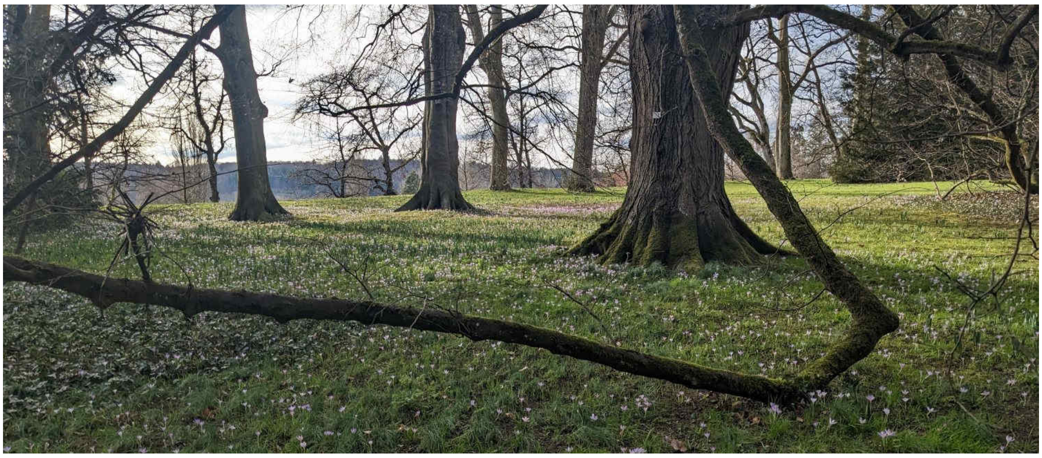


Island of Mainau (by Philipp di Dio, 18th of February 2023)

1 Invited Speaker

- Hamza Fawzi (University of Cambridge)
- Jean-Bernard Lasserre (LAAS-CNRS Toulouse)
- Monique Laurent (CWI Amsterdam and Tilburg University)
- Tim Netzer (University of Innsbruck)
- Konrad Schmüdgen (University of Leipzig)
- Markus Schweighofer (University of Konstanz)
- Jan Stochel (Jagiellonian University, Krakow)

2 About the Organizers

2.1 Philipp J. di Dio (Main Organizer)

Philipp di Dio is a 5 Year Research Fellow (research group leader) at the Zukunftskolleg (Institute for Advanced Studies) with his own group since 2022. He studied chemistry and mathematics at the second oldest German university, the University of Leipzig founded 1409. After his bachelor in chemistry and his diploma in mathematics he did his PhD under supervision of Konrad Schmüdgen on the moment problem as a member of the Max Planck Institute for Mathematics in the Science in Leipzig with a PhD-scholar ship. He did a PostDoc at the TU Berlin under supervision of Mario Kummer and at the LAAS-CNRS in Toulouse under supervision of e.g. Jean-Bernard Lasserre. With his DFG grant he joined in 2022 the University of Konstanz and the Zukunftskolleg where he works on the interplay between moments, polynomials, and partial differential equations.

2.2 Tobias Sutter (Co-Organizer)

Tobias Sutter is tenure-track-professor for Computer Science with focus on Machine Learning at the University of Konstanz since 2021. Since 2022 he is a member of the Centre for Human — Data — Society. He received his bachelor's degree in 2007 on a thesis of Mechanical Engineering at the ETH Zürich (Switzerland), followed by the master's degree in 2012. In 2017 he earned a PhD in Electrical Engineering (thesis: Convex programming in optimal control and information theory). After postdoctoral research positions at EPF Lausanne and ETH Zürich he is now Fellow of the Institute for Advanced Study at the University of Konstanz. He is Group leader at the Cluster of Excellence: Centre for the Advanced Study of Collective Behaviour and Program Committee of the Annual Learning for Dynamics & Control Conference (L4DC). His research is focused on Machine Learning, Stochastic Optimization and Control and Information Theory.

3 Participants

1) Baldi, Lorenzo (Max Planck Institute for Mathematics in the Sciences, Leipzig)
2) Blomenhofer, Alexander Taveira (CWI Amsterdam)
3) Brüser, Clemens (TU Dresden)
4) Brosch, Daniel (University of Klagenfurt)
5) Cimpric, Jaka (University of Ljubljana)
6) di Dio, Philipp (University of Konstanz)
7) Eggen, Carl (University of Konstanz)
8) El Azhar, Hamza (Chouaib Doukkali University, Morocco)
9) van der Eyden, Mirte (University of Innsbruck)
10) Fawzi, Hamza (University of Cambridge)
11) Garloff, Jürgen (University of Konstanz and HTWK Konstanz)
12) Grunwald, Rüdiger (University of Konstanz)
13) Nevado, Alejandro González (University of Konstanz)
14) Gutschlag, Theodor (University of Konstanz)
15) Hess, Sarah-Tanja (University of Konstanz)
16) Jablonski, Zenon (Jagiellonian University, Krakow)
17) Kaiser, Marcel (University of Konstanz)
18) Klingler, Andreas (University of Innsbruck)
19) Kovalyov, Ivan (University of Osnabrück)
20) Kozhasov, Khazhgali (Université Côte d'Azur, Nice)
21) Krapp, Lothar Sebastian (University of Konstanz)
22) Langer, Lars-Luca (University of Konstanz)
23) Lasserre, Jean-Bernard (LAAS-CNRS Toulouse)
24) Laurent, Monique (CWI Amsterdam and Tilburg University)
25) Metzlaff, Tobias (University of Kaiserslautern-Landau)

26) Michalek, Mateusz (University of Konstanz)
27) Miller, Jared (ETH Zurich)
28) Netzer, Tim (University of Innsbruck)
29) Pietrzycki, Pawel (Jagiellonian University, Krakow)
30) Ríos-Zertuche, Rodolfo (UiT The Arctic University of Norway, Tromsø)
31) Salac, Radek (University of Konstanz)
32) Sawall, David (University of Konstanz)
33) Schick, Moritz (University of Konstanz)
34) Schmüdgen, Konrad (University of Leipzig)
35) Schötz, Matthias (Polish Academy of Sciences)
36) Schweighofer, Markus (University of Konstanz)
37) Slot, Lucas (ETH Zurich)
38) Stochel, Jan (Jagiellonian University, Krakow)
39) Sun, Shengding (University of Cambridge)
40) Sutter, Tobias (University of Konstanz)
41) Vargas, Luis Felipe (IDSIA Lugano)
42) Zalar, Aljaž (University of Ljubljana)

Island of Mainau (by Philipp di Dio, 22nd of April 2023)

4 Talks and Abstracts

4.1 Baldi, Lorenzo: Carathéodory numbers and flat extension for measures supported on genus one curves

joint work with Greg Blekherman and Rainer Sinn

After effectively describing the convex cone of nonnegative homogeneous polynomials on projective, algebraic curves, we focus on the genus one case and investigate the dual moment problem. As a guiding example, we consider the case of plane cubics.

We exactly determine the Carathéodory number of the moment cone (truncated in any even degree) on such genus one projective curves and show that it depends on the topology of the real locus. As a consequence, we are able to solve the truncated moment problem in the affine case, introducing the notion of almost flat extension.

4.2 Blomenhofer, Alexander Taveira: Nondefectivity of invariant secant varieties

joint work with Alex Casarotti

We study the dimensions of higher-order secants of varieties, which live in a G-module and are invariant under the action of the group G. We show that the secant dimensions behave as expected, up to a few potential exceptions. As an application, we partially resolve a conjecture due to Baur, Draisma and de Graaf on Grassmannian varieties.

4.3 Brüser, Clemens: Quadratic Determinantal Representations of Positive Polynomials

joint work with Mario Kummer

Given a projective plane curve $C = V(f)$ of even degree, a symmetric quadratic determinantal representation of f is a symmetric matrix M, all entries of which are homogeneous polynomials of degree 2, satisfying $\det(M) = f$. If f is defined over the real numbers and $M(a)$ is psd for every triple of real numbers a, then we call such a representation positive semi-definite.

It has been shown before that there exist non-negative polynomials f that do not admit such a psd representation. I will derive this result using a new approach, which I will also successfully apply to the Robinson Polynomial - an example for which an answer had been unknown so far. This confirms a conjecture by Buckley and Sivic.

I will then restrict considerations to plane quartic curves. It is well-known that there are exactly 28 bitangent lines to every given smooth quartic curve C - a result, which has previously been applied to determine all its linear determinantal representations. Using the combinatorial structure of the bitangents (interpreted as the set of odd theta characteristics of C), I will build on this existing work to prove that for every smooth quartic curve C there exist exactly 4 psd quadratic representations. Furthermore I will relate these representations to the real-valued points of the Jacobian of C.

4.4 Brosch, Daniel: Combinatoric derivations in extremal graph theory and Sidorenko's conjecture

Sidorenko's conjecture can be formulated as

> Let H be a bipartite graph, and $\rho \in [0, 1]$. Of all the graphs with edge density ρ, the graph(-limit) obtained by picking edges uniformly at random minimizes the homomorphism density of H.

This conjecture, first formulated in 1991 by Sidorenko, has received considerable attention over the last decades, and yet remains open in the general case.

It was shown recently [Blekherman, Raymond, Singh, Thomas, 2020] that sums-of-squares in Razborov's flag algebra are not strong enough to prove even small, known cases of the conjecture. To circumvent this, we introduce a novel kind of (Lie-)derivation of flags. Due to their combinatoric nature, we can use them to systematically gain knowledge on global minimizers of problems in extremal graph theory. We combine them with the flag algebra method to find new proofs for various cases of Sidorenko's conjecture.

4.5 Eggen, Carl: Granularity for mixed-integer polynomial optimization problems

joint work with Oliver Stein and Stefan Volkwein

Finding good feasible points is crucial in mixed-integer programming. For this purpose we combine a sufficient condition for consistency, called granularity, with the moment-/sos-hierarchy from polynomial optimization. If the mixed-integer problem is granular, we get feasible points by solving continuous polynomial problems and rounding their optimal solutions. The moment-/sos-hierarchy is hereby used to calculate those continuous polynomial problems. Numerical examples will illustrate our method.

4.6 van der Eyden, Mirte: Beyond free spectrahedra

A spectrahedron is a solution of a linear matrix inequality. We can make this 'free' by considering the same inequality for matrices of all sizes simultaneously. Free spectrahedra are based on the notion of positive semidefinitene matrices and are examples of operator systems, which connect operator algebra, free semialgebraic geometry and quantum information theory. In this work we generalize operator systems and many of their theorems. While positive semidefinite matrices form the underlying structure of operator systems, our work shows that these can be promoted to far more general structures. For instance, we prove a general extension theorem which unifies the well-known homomorphism theorem, Riesz' extension theorem, Farkas' lemma and Arveson's extension theorem. On the other hand, the same theorem gives rise to new vector-valued extension theorems, even for invariant maps, when applied to other underlying structures. We also prove generalized versions of the Choi-Kraus representation, Choi-Effros theorem, duality of operator systems, factorizations of completely positive maps, and more, leading to new results even for operator systems themselves.

4.7 El Azhar, Hamza: On the square root problem

For recursively generated shifts, we provide definitive answers to two outstanding problems in the theory of unilateral weighted shifts: the Subnormality Problem (SP) (related to the Aluthge transform) and the Square Root Problem (SRP) (which deals with Berger measures of subnormal shifts). We use the Mellin Transform and the theory of exponential polynomials to establish that (SP) and (SRP) are equivalent if and only if a natural functional equation holds for the canonically associated Mellin transform. For p–atomic measures with $p \leq 6$, our main result provides a new and simple proof of the above mentioned equivalence. Subsequently, we obtain an example of a 7-atomic measure for which the equivalence fails. This provides a negative answer to a problem posed by G. R. Exner in 2009, and to a recent conjecture formulated by R. E. Curto et. al. in 2019.

4.8 Fawzi, Hamza: Ground State Optimization

4.9 Jablonski, Zenon: Subnormality via moments

joint work with Piotr Budzynski, Il Bong Jung and Jan Stochel

The celebrated Lambert characterization of subnormality states that a (closed) bounded linear operator is subnormal if and only if it generates Stieltjes moment sequences. During the talk we will discuss two problems:

1) Is it true that there exists a non-hyponormal injective operator that generates Stieltjes moment sequences?
   The question of how simple such an example can be will be discussed.

2) Is it true that for every positive integer n, there exists a subnormal operator whose nth power is densely defined and the domain of its $(n+1)$th power is trivial?

4.10 Klingler, Andreas: A homotopy method for convex optimization

joint work with Tim Netzer

Convex optimization concerns the problem of finding the maximum of a linear function over a convex set. This class covers many optimization problems in quantum information, portfolio optimization, and machine learning. A common strategy to solve these problems is the so-called interior point method, which works well for subclasses like semidefinite or geometric programs.

In this talk, we will introduce a new approach to solving convex optimization problems via a homotopic approach. In this approach, we deform an optimization problem with a trivial solution into the target problem and keep track of the solutions along the homotopy. This approach is motivated by the field of numerical algebraic geometry, which solves systems of polynomial equations using a similar idea.

Our method applies to certain convex optimization problems, including Semidefinite Programs, Hyperbolic Programs, and convex optimization problems with a single convexity constraint. Moreover, we will show several benchmark problems in which this method outperforms known lifting techniques to semidefinite programs.

4.11 Kovalyov, Ivan: A truncated indefinite Stieltjes moment problem

A classical Stieltjes moment problem was studied in [8]. Given a sequence of real numbers $\mathbf{s} = \{s_i\}_{i=0}^{\infty}$, find a positive Borel measure σ with a support on $\mathbb{R}_+ = [0, \infty)$, such that

$$\int_{\mathbb{R}_+} t^j d\sigma(t) = s_j \qquad (j \in \mathbb{Z}_+). \tag{1}$$

By the Hamburger–Nevanlinna theorem [1], the truncated Stieltjes moment problem can be reformulated in terms of the Stieltjes transform

$$f(z) = \int_{\mathbb{R}_+} \frac{d\sigma(t)}{t - z} \qquad z \in \mathbb{C} \backslash \mathbb{R}_+ \tag{2}$$

of σ as the following interpolation problem at ∞

$$f(z) = -\frac{s_0}{z} - \frac{s_1}{z^2} - \cdots - \frac{s_{2n}}{z^{2n+1}} + o\left(\frac{1}{z^{2n+1}}\right), \qquad z \widehat{\to} \infty. \tag{3}$$

The notation $z \widehat{\to} \infty$ means that $z \to \infty$ nontangentially, that is inside the sector $\varepsilon < \arg z < \pi - \varepsilon$ for some $\varepsilon > 0$.

In the current talk, we consider an indefinite truncated Stieltjes moment problem in the generalized Stieltjes class $\mathbf{N}_\kappa^k$. Let us recall:

A function f meromorphic on $\mathbb{C} \backslash \mathbb{R}$ with the set of holomorphy $\mathfrak{h}_f$ is said to be in the generalized Nevanlinna class $\mathbf{N}_\kappa$ ($\kappa \in \mathbb{N}$), if for every set $z_i \in \mathbb{C}_+ \cap \mathfrak{h}_f$ ($j = 1, \ldots, n$) the form

$$\sum_{i,j=1}^{n} \frac{f(z_i) - \overline{f(z_j)}}{z_i - \overline{z}_j} \xi_i \overline{\xi}_j$$

has at most κ and for some choice of z_i ($i = 1, \ldots, n$) it has exactly κ negative squares. For $f \in \mathbf{N}_\kappa$ let us write $\kappa_-(f) = \kappa$. In particular, if $\kappa = 0$ then the class $\mathbf{N}_0$ coincides with the class $\mathbf{N}$ of Nevanlinna functions.

A function $f \in \mathbf{N}_\kappa$ is said to belong to the class $\mathbf{N}_\kappa^k$ ($k \in \mathbb{N}$) if $zf \in \mathbf{N}_\kappa^k$ (see [2]). If $\kappa = k = 0$ the class $\mathbf{N}_0^0$ coincides with the Stieltjes class introduced by M.G. Krein in 1952.

Problem $\mathbf{MP}_\kappa^k(\mathbf{s}, \ell)$. Given ℓ, κ, $k \in \mathbb{Z}_+$, and a sequence $\mathbf{s} = \{s_i\}_{i=0}^{\ell}$ of real numbers, describe the set $\mathcal{M}_\kappa^k(\mathbf{s})$ of functions $f \in \mathbf{N}_\kappa^k$, which have the following asymptotic expansion

$$f(z) = -\frac{s_0}{z^1} - \frac{s_1}{z^2} - \cdots - \frac{s_\ell}{z^{\ell+1}} + o\left(\frac{1}{z^{\ell+1}}\right), \quad z\widehat{\rightarrow}\infty. \tag{4}$$

The description of the solution is found in terms of the continues fractions and generalized Stieltjes polynomials [3]–[6].

[1] N. I. Akhiezer, The classical moment problem, Oliver and Boyd, Edinburgh, 1965.

[2] V. Derkach. *On Weyl function and generalized resolvents of a Hermitian operator in a Kreĭn space.* Integral Equations Operator Theory, **23**, 387-415 (1995).

[3] V. Derkach, I. Kovalyov, *On a class of generalized Stieltjes continued fractions*, Methods of Funct. Anal. and Topology, **21**(4) (2015), 315-335.

[4] V. Derkach, I. Kovalyov, *The Schur algorithm for indefinite Stieltjes moment problem*, Math. Nachr. (2017) DOI: 10.1002/mana.201600189.

[5] I. Kovalyov, *A truncated indefinite Stieltjes moment problem*, J. Math. Sci. **224** (2017) 509–529.

[6] I. Kovalyov, *Regularization of the indefinite Stieltjes moment problem*, Linear Algebra Appl. **594** (2020) 1–28.

[7] M. G. Kreĭn and H. Langer. *Über einige Fortsetzungsprobleme, die eng mit der Theorie Hermitscher Operatoren in Raume π_κ zusammenhangen, II. Verallgemeinerte Resolventen u-Resolventen und ganze Operatoren*, J. Funct. Anal. **30**, 390-447 (1978).

[8] T. J. Stieltjes, *Recherches sur les fractions continues*, Ann. Fac. Sci. de Toulouse **8**, 1–122 (1894).

4.12 Kozhasov, Khazhgali: When are odd powers of non-negative forms SOS?

joint work with Greg Blekherman, Pablo Parrilo, Bruce Reznick, and Isabelle Shankar

Given a nonnegative form. A natural question is to understand whether its k-th power (for some odd k) is a sum of squares. In the talk I will discuss this problem, focusing on the case of ternary sextics.

4.13 Krapp, Lothar Sebastian: Schanuel's Conjecture, Roots of Exponential Polynomials and Peano Arithmetic

joint work with Merlin Carl

An exponential polynomial in n variables is an expression of the form

$$p(x_1, \dots, x_n, 2^{x_1}, \dots, 2^{x_n}),$$

where $p(x_1, \dots, x_n, y_1, \dots, y_n)$ is a polynomial with integer coefficients in $2n$ variables. It is not yet known if there exists a procedure that determines whether a given exponential polynomial has a real root. In fact, due to the work of Macintyre and Wilkie [2], if such a procedure exists, then the complete first-order theory of the real exponential field $(\mathbb{R}, +, \cdot, 2^x)$ is decidable. Moreover, if Schanuel's Conjecture – a famous open result from Transcendental Number Theory – is true, then such a procedure exists. In my talk, I will illustrate how the basic axioms from Peano Arithmetic can potentially be used to (effectively) determine whether a given exponential polynomial has a real root. Again, one of the main obstacles is Schanuel's Conjecture. All relevant background from Logic will briefly be introduced. This is joint work with Merlin Carl [1].

[1] M. Carl, L. S. Krapp, Models of true arithmetic are integer parts of models of real exponentiation, *J. Log. Anal.* **13:3** (2021) 1–21.

[2] A. Macintyre, A. J. Wilkie, On the decidability of the real exponential field, Kreiseliana: about and around Georg Kreisel, ed. P. Odifreddi; A. K. Peters, Wellesley, MA (1996), 441–467.

4.14 Lasserre, Jean-Bernard: The Christoffel function: New applications, connections and extensions

We provide an introduction to the Christoffel function (CF), a well-known (and old) tool from the theory of approximation and orthogonal polynomials. We then describe how it provides a simple and easy-to-use tool to address some problems in data analysis and mining, and in approximation and interpolation of discontinuous functions. Finally we also reveal some surprising links of the CF with seemingly different disciplines, including polynomial optimization, positivity certificates, and equilibrium measures of compact sets. At last but not least we also provide a modified version of the Christoffel function with a nice asymptotic property.

4.15 Laurent, Monique: Sums of Squares of Polynomials for Matrix Copositivity and Stable Sets in Graphs

joint work with Luis Felipe Vargas

This lecture revolves around the question of testing matrix copositivity, which can be formulated as testing nonnegativity of a quadratic form (over the nonnegative orthant or the simplex), or of a quartic form (on the full space or the unit sphere). Various sum-of-squares positivity certificates lead to various conic approximations of the copositive cone. We discuss old and new results about these conic approximations, their mutual relationships, and when they are sufficient to fully describe copositive matrices. We also discuss their application to finding bounds for stable sets in graphs and finite convergence properties of these bounds.

4.16 Metzlaff, Tobias: Chebyshev Moments

joint work with Evelyne Hubert, Philippe Moustrou, and Cordian Riener

When optimizing a symmetric trigonometric polynomial, it is possible and convenient to rewrite the problem in terms of classical polynomials on a basic semi-algebraic set and to apply Lasserre's hierarchy. A suitable basis turns out to be the basis of generalized Chebyshev polynomials instead of the standard monomial one. We explore the behavior of matrix moments and matrix sums of squares in this basis and provide a flat extension type criterion.

Associated paper:

[1] HAL-03768067

4.17 Michalek, Mateusz: Enumerative geometry meets statistics and optimization

We build on recent developments by June Huh relating combinatorial invariants of matroids to algebraic geometry. We propose a theory where to a linear space of matrices we associate naturally a cohomology class of a very nice variety (e.g. variety of complete collineations). Numerical invariants of this class recover among others chromatic polynomials of graphs and matroids, characteristic numbers of quadrics, maximum likelihood degree of linear concentration models, algebraic degree of semidefinite programming and Euler characteristic of determinental hypersurfaces.

4.18 Miller, Jared: Maximizing Slice-Volumes of Semialgebraic Sets using Sum-of-Squares Programming

joint work with Chiara Meroni, Matteo Tacchi, and Mauricio Velasco

We presents an algorithm to maximize the volume of an affine slice through a given semi-algebraic set. This slice-volume task is formulated as an infinite-dimensional linear program in continuous functions, inspired by prior work in volume computation of semialgebraic sets. A convergent sequence of upper-bounds to the maximal slice volume are computed using the moment-Sum-of-Squares hierarchy of semidefinite programs in increasing size. The computational complexity of this scheme can be reduced by utilizing topological structure (in dimensions 2, 3, 4, 8) and symmetry. This numerical convergence can be accelerated through the introduction of redundant Stokes-based constraints. Demonstrations of slice-volume calculation are performed on example sets.

4.19 Netzer, Tim: Zeros of Polynomial Equations in Matrices, Quaternions and Beyond

joint work with Maximilian Illmer and Andreas Thom

Every non-constant complex polynomial has a complex zero, and every real polynomial of odd degree has a real zero. How about other, and in particular non-commutative algebras? We prove a general result about existence of zeros of polynomials over algebras, in a complex and a real version. The result applies for example to (Hermitian) matrices, quaternions and octonions. For quaternions, this strengthens a known fundamental theorem on zeros, and we also provide an example, based on sums of squares, that it cannot be strengthened further.

4.20 Pietrzycki, Pawel: Operator moments and Arveson's hyperrigidity

joint work with Jan Stochel

One of the features of a Borel spectral measure on $\mathbb{R}$ is the multiplicativity of the corresponding Stone-von Neumann functional calculus. In particular, if E is a Borel spectral measure on $\mathbb{R}$ with compact support, then the following identities hold

$$\Big(\int_{\mathbb{R}} xE(\mathrm{d}x)\Big)^n = \int_{\mathbb{R}} x^n E(\mathrm{d}x), \quad n = 1, 2, \dots. \tag{5}$$

Hence, all operator moments of E are determined by the first one, and according to the spectral theorem there is a one-to-one correspondence between Borel spectral measures on $\mathbb{R}$ and their first operator moments. This is no longer true for general Borel semispectral measures on $\mathbb{R}$. It turns out, however, that the single equality in (5) with $n = 2$ guarantees spectrality.

It turns out that, from a mathematical and physical point of view, it is important to investigate the relationship between semispectral and spectral measures. In the classical von Neumann description of quantum mechanics selfadjoint operators or, equivalently, Borel spectral measures on the real line represent observables. This approach is insufficient in describing many natural properties of measurements, such as measurement inaccuracy. Therefore, in standard modern quantum theory, the generalization to semispectral measures is widely used.

In 2006 Kiukas, Lahti and Ylinen asked the following general question. *When is a positive operator measure projection valued?* A version of this question formulated in terms of operator moments was posed in [1]. *Let T be a selfadjoint operator and F be a Borel semispectral measure on the real line with compact support. For which positive integers $p < q$ do the equalities $T^k = \int_{\mathbb{R}} x^k F(\mathrm{d}x)$, $k = p, q$, imply that F is a spectral measure?* In this talk, we show that the answer is affirmative if p is odd and q is even, and negative otherwise.

Motivated both by the fundamental role of the classical Choquet boundary in classical approximation theory, and by the importance of approximation in the contemporary theory of operator algebras, Arveson introduced hyperrigidity as a form of approximation that captures many important operator-algebraic phenomena. We discuss the relationship between hyperrigidity and the aforementioned spectral measure characterization.

[1] P. Pietrzycki, J. Stochel, Subnormal nth roots of quasinormal operators are quasinormal, *J. Funct. Anal.* **280** (2021), 109001.

[2] P. Pietrzycki, J. Stochel, Two-moment characterization of spectral measures on the real line, *Canad. J. Math.* vol. 75 (4) (2023), 1369-1392

[3] P. Pietrzycki, J. Stochel, Hyperrigidity, operator moments and convergence of subnormal operators, in preparation.

4.21 Ríos-Zertuche, Rodolfo: Relaxation gaps for SDP-based algorithms for PDEs and optimal control

joint work with Didier Henrion, Milan Korda, and Martin Kružík

The occupation measure relaxation provides a way to convert a large family of partial differential equations and optimal control problems into semidefinite programs. In this way, it leads to algorithms that render those problems tractable and open the door to a broad spectrum of applications. This is why the question of a gap between the classical results and those obtained through the occupation measure relaxation has spiked substantial interest. I will discuss recent results regarding the absence and presence of such gaps in different contexts, including both convex and nonconvex problems, and I will also discuss the meaning of those gaps to the problems in question.

Associated papers:

[1] ArXiv2303.14824

[2] ArXiv2303.02434

[3] ArXiv2205.14132

4.22 Sawall, David: Amalgamation of real zero polynomials with two common variables

joint work with Mario Kummer

Whenever we get a polynomial form another polynomial by substituting zero for some of the variables, we call the second polynomial an *extension* of the first one. A polynomial is called *real zero* if it has only real zeros along all lines through the origin and does not vanish at the origin. We will show that there are two real zero polynomials with only two common variables such that there is no common extension (called *amalgam*) which is again real zero. We derive this example from two polymatroids which cannot be written as restrictions of a larger polymatroid.

4.23 Schick, Moritz: Quantitative relation of the cones of sums of squares and sums of nonnegative circuit polynomials

joint work with Mareike Dressler and Salma Kuhlmann

An n-variate homogeneous polynomial (form) is positive semidefinite (PSD), if it attains nonnegative values over the whole $\mathbb{R}^n$. The set of all such forms of fixed degree $2d$ is a convex cone, called the PSD cone. Studying subcones of the PSD cone is a rich topic in Real Algebraic Geometry and can be applied to polynomial optimization problems. Two subcones of the PSD cone, which are studied extensively in the literature are the sums of squares (SOS) cone and the sums of nonnegative circuit polynomials (SONC) cone. In this talk, we formally introduce the three cones, and study their set-theoretic relation. We illustrate different methods to find forms separating two of the cones, respectively. In particular, it was recently shown by Reznick, that the odd powers of the Motzkin form M are not SOS. In this context, we show that M^k is SONC if and only if $k = 1$.

4.24 Schötz, Matthias: Real Nullstellensatz for 2-step nilpotent Lie algebras

joint work with Philipp Schmitt

For a finite-dimensional real Lie algebra g the universal enveloping algebra of the complexifiction of g is a non-commutative, finitely generated complex $*$-algebra $U^*(g)$. For any such g one can ask whether the following real Nullstellensatz holds:

> An ideal I of $U^*(g)$ is the kernel of an integrable $*$-representation of $U^*(g)$ if and only if I is a real $*$-ideal.

The integrable $*$-representations in this statement are those $*$-representations of $U^*(g)$ that are obtained as the derivative of a strongly continuous unitary representation of the connected and simply connect Lie group that integrates g. In the special case that g is an abelian Lie algebra, the real Nullstellensatz for g holds – this is simply the classical, commutative real Nullstellensatz.

In this talk I will present the real Nullstellensatz for 2-step nilpotent real Lie algebras. While such Lie algebras are rather trivial from the point of view of Lie theory, their real

Nullstellensatz is not trivial at all because the irreducible integrable $*$-representations are either 1-dimensional (i.e. complex-valued unital $*$-homomorphisms) or unbounded, and both types need to be taken into account.

4.25 Schmüdgen, Konrad: The matricial truncated moment problem

Suppose $\mathcal{E}$ is a finite-dimensional real vector space of measurable mappings of a measurable space $(\mathcal{X}, \mathfrak{X})$ into the complex Hermitean $q \times q$-matrices $\mathcal{H}_q$, $q \in \mathbb{N}$. A linear functional Λ on $\mathcal{E}$ is called a *moment functional* if there exists a positive $\mathcal{H}_q$-valued measure μ on $(\mathcal{X}, \mathfrak{X})$ such that

$$\Lambda(F) = \int \langle F, \mathrm{d}\mu \rangle \quad \text{for all} \quad F \in \mathcal{E}.$$

In this talk matricial versions of basic results and notions on the truncated moment problem are given. A particular emphasis is on the peculiarities of the matrix case.

Matricial versions of the Richter-Tchakaloff theorem and the flat extension theorem are developed. It is shown that strictly positive linear functionals are moment functionals. The class of commutative moment functionals and the set of masses of representing measures at a fixed point are investigated. For a moment functional Λ, the definition of the core set $\mathcal{V}(\Lambda)$ is much more subtle than in the scalar case. We propose a definition of $\mathcal{V}(\Lambda)$ and obtain as a main result that the core set $\mathcal{V}(\Lambda)$ is equal to the set of atoms $\mathcal{W}(\Lambda)$.

4.26 Schweighofer, Markus: Pure states for polynomial nonnegativity certificates in the presence of zeros

joint work with Luis Vargas

We recently gave new characterizations of copositive matrices of size five and of the stability number of a graph in terms of polynomial sum-of-squares representations. These are just two examples of the largely unexplored potential of pure states in the theory of sum-of-squares representations of polynomials with (potentially infinitely many) zeros. This theory has been started in 2012 by Burgdorf, Scheiderer and myself but still is not commonly known. This tutorial talk is primarily meant for those who have never heard about pure states.

4.27 Slot, Lucas: Rational Christoffel functions for measures with conditional independence

joint work with Jean-Bernard Lasserre

Christoffel polynomials are a tool from approximation theory, associated with a measure μ on $\mathbb{R}^d$ and indexed by their degree n. They can be used to approximate the (compact) support of μ, or even its density as $n \to \infty$. Their coefficients are computed by inverting the (truncated) moment matrix of μ, which has size $\binom{n+d}{n}$. Practically, this is prohibitive for larger values of d. We introduce a variation of the Christoffel polynomial for measures μ with conditional independence structure, represented by a graphical model G. Its coefficients can be computed by inverting at most d matrices of size $(n+1)^{t+1}$, where t is the treewidth of G. For instance, if μ is a product measure, $t(G) = 0$. It is thus significantly easier to compute when $d \gg t(G)$. Nevertheless, we show that our variant retains many of the standard Christoffel polynomial's desirable properties.

4.28 Stochel, Jan: Moment problem for discrete measures via idempotents

joint work with Hamza El-Azhar and Ayoub Harrat

My goal will be to present the results that we obtained together with Hamza El-Azhar and Ayoub Harrat [1] by examining the full multidimensional moment problem for discrete measures using Vasilescu's approach via idempotents with respect to the associated (positive) Riesz functional Λ. In particular, I will give a sufficient condition for the existence of a discrete integral representation of the Riesz functional Λ, which turns out to be necessary in the bounded shift space case (for the necessity, it suffices to assume the density of polynomials in the corresponding L^2-space). The necessary and sufficient conditions are written in terms of Λ-multiplicative idempotents. I will pay special attention to characterize Λ-multiplicative elements by providing several criteria guaranteeing that they are characteristic functions of single point sets. We also give an example showing that in general Λ-multiplicative elements may not be characteristic functions of single point sets.

[1] Hamza El-Azhar, Ayoub Harrat, Jan Stochel, Idempotents and moment problem for discrete measure, *Linear Algebra and its Applications* **628** (2021), 202-227.

4.29 Sun, Shengding: Generalizing determinantal inequalities using linear prinicipal minor (lpm) polynomials

joint work with Greg Blekherman, Mario Kummer, Raman Sanyal and Kevin Shu

Linear principal minor (lpm) polynomials are linear combinations of principal minors of a symmetric matrix. PSD stable lpm polynomials have one-to-one correspondence with stable multiaffine polynomials (from the work by Petter Branden on stability preservers), and we show that the classical Hadamard-Fischer determinantal inequalities have natural generalizations for PSD stable lpm polynomials. Using this construction we also show the existence of a hyperbolic polynomial (degree 3 in 6 variables) some of whose Rayleigh differences are not sums of squares.

4.30 Vargas, Luis Filipe: Complexity Results About the Exactness of Sum-of-Squares Approximations for Polynomial Optimization

We study polynomial optimization problems for which the Lasserre sum-of-squares hierarchy has finite convergence. Nie (2012) shows that if the problem has finitely many minimizers and they satisfy the classical optimality conditions, then finite convergence holds. Using this, he shows that finite convergence holds generically. In this talk we show that 1) testing whether a standard quadratic program has finitely many minimizers is NP-hard. Moreover, 2) testing whether the Lasserre hierarchy of a polynomial optimization has finite convergence is NP-hard. We finish with a discussion on new methods for showing finite convergence in the presence of infinitely many minimizers.

4.31 Zalar, Aljaž: A gap between positive even quartics and sums of squares ones

joint work with Igor Klep and Tea Štrekelj

In the talk we will present estimates for the ratio between the volume radii of suitable compact sections of the cone of positive even quartics and the cone of sums of squares even quartics. The estimates show that the difference between the cones in question does not become arbitrarily large as the number of variables grows to infinity, which is in sharp contrast to the case of all quartic forms established by Blekherman. Moreover, a free probability based construction for generating positive, but not sums of squares even quartics will be presented.

5 Program

5.1 Monday, March 11th 2024

Room K503

10.00–10.20 *Arrival*

10.25–10.30 Philipp di Dio and Tobias Sutter
Greetings and Opening

10.35–11.20 Konrad Schmüdgen
The matricial truncated moment problem

11.25–11.55 Pawel Pietrzycki
Operator moments and Arveson's hyperrigidity

12.00–13.30 *Lunch and Coffee*

13.30–14.15 Monique Laurent
Sums of Squares of Polynomials for Matrix Copositivity and Stable Sets in Graphs

14.20–15.05 Markus Schweighofer
Pure states for polynomial nonnegativity certificates in the presence of zeros

15.10–15.30 *Coffee*

15.30–15.55 Luis Filipe Vargas
Complexity Results About the Exactness of Sum-of-Squares Approximations for Polynomial Optimization

16.00–16.25 Mirte van der Eyden
Beyond free spectrahedra

16.30–16.55 Aljaž Zalar
A gap between positive even quartics and sums of squares ones

5.2 Tuesday, March 12th 2024

Room K503

9.00– 9.45 Jan Stochel
Moment problem for discrete measures via idempotents

9.50–10.20 Matthias Schötz
Real Nullstellensatz for 2-step nilpotent Lie algebras

10.25–10.45 *Coffee*

10.45–11.10 Lorenzo Baldi
Carathéodory numbers and flat extension for measures supported on algebraic curves

11.15–11.35 Moritz Schick
Quantitative relation of the cones of sums of squares and sums of nonnegative circuit polynomials

11.40–12.00 Lothar Sebastian Krapp
Schanuel's Conjecture, Roots of Exponential Polynomials and Peano Arithmetic

12.00–13.00 *Lunch*

13.00–17.00 ***Excursion to Hohentwiel*** (see Section 6)

5.3 Wednesday, March 13th 2024

Room K503

9.00– 9.45 Jean-Bernard Lasserre
The Christoffel function: New applications, connections and extensions

9.50–10.20 Lucas Slot
Rational Christoffel functions for measures with conditional independence

10.25–10.45 *Coffee*

10.45–11.10 Rodolfo Ríos-Zertuche
Relaxation gaps for SDP-based algorithms for PDEs and optimal control

11.15–11.35 Tobias Metzlaff
Chebyshev Moments

11.40–12.00 Jared Miller
Maximizing Slice-Volumes of Semialgebraic Sets using Sum-of-Squares Programming

12.05–13.00 *Lunch and Coffee*

13.00–13.45 Hamza Fawzi
TBA

13.50–14.15 Shengding Sun
Generalizing determinantal inequalities using linear prinicipal minor (lpm) polynomials

14.20–14.45 Andreas Klingler
A homotopy method for convex optimization

14.50–15.10 *Coffee*

15.10–15.30 Carl Eggen
Granularity for mixed-integer polynomial optimization problems

15.35–15.55 David Sawall
Amalgamation of real zero polynomials with two common variables

16.00–24.00 ***Visiting Mainau and Conference Dinner*** (see Section 7)

5.4 Thursday, March 14th 2024

Room K503

9.00– 9.45 Tim Netzer
Zeros of Polynomial Equations in Matrices, Quaternions and Beyond

9.50–10.20 Zenon Jablonski
Subnormality via moments

10.25–10.45 *Coffee*

10.45–11.10 Khazhgali Kozhasov
When are odd powers of non-negative forms SOS?

11.15–11.40 Alexander Taveira Blomenhofer
Nondefectivity of invariant secant varieties

11.45–12.05 Ivan Kovalyov
A truncated indefinite Stieltjes moment problem

12.05–13.00 *Lunch and Coffee*

13.00–13.30 Mateusz Michalek
Enumerative geometry meets statistics and optimization

13.35–14.00 Daniel Brosch
Combinatoric derivations in extremal graph theory and Sidorenko's conjecture

14.05–14.30 Hamza El Azhar
On the square root problem

14.35–14.55 Clemens Brüser
Quadratic Determinantal Representations of Positive Polynomials

15.00–15.30 Philipp di Dio and Tobias Sutter
Farewell and Coffee

6 Excursion to Hohentwiel

Hohentwiel (by Philipp di Dio, 25th of August 2022)

The excursion will take place on Tuesday, 12th of March 2024. We will visit the Hohentwiel Fortress Ruins

https://www.festungsruine-hohentwiel.de

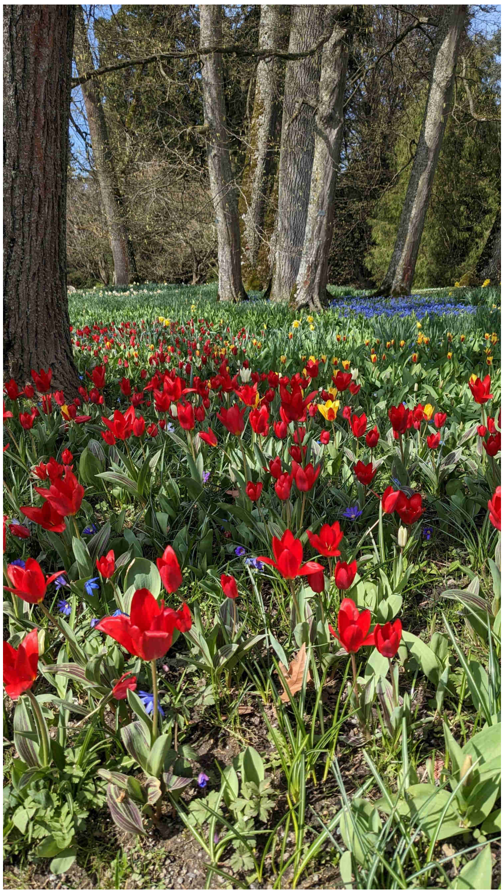
with a beautiful view over the landscape. We plan to leave the University of Konstanz at 13.00 by bus and return at 17.00 from Hohentwiel.

The participation fee will be 20€ which includes entrance to Hohentwiel and transportation by bus from the University of Konstanz to Hohentwiel and back.

In case of bad weather forecast an alternative destination will be planned.

Hohentwiel (by Philipp di Dio, 25th of August 2022)

7 Conference Dinner at the Island of Mainau



Island of Mainau (by Philipp di Dio, 28th of March 2023)

The conference dinner will take place Wednesday evening, 13th of March 2023, at the Island of Mainau

https://www.mainau.de

at the Restaurant Comturey.

We plan to finish with the talks at approx. 16.00 to walk from the university to the island of Mainau (2km, 30min). A taxi can be organized going from the university to the island, so please let us know in advance if you can not walk this far.

The dinner will start at 18.00. The dinner fee will be 40€ which already includes the entrance fee to the island and drinks at the dinner. The island and the dinner venue will close for us at 24.00.

After the dinner the regular bus service can be used to go to the city/hotels.

Island of Mainau (by Philipp di Dio, 28th of March 2023)

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Universität Konstanz

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Sunrise over Lake Konstanz (by Philipp di Dio, 7th of February 2023, 7:25)